

DIAGONALISABLE p -GROUPS ACTING ON PROJECTIVE VARIETIES CANNOT FIX EXACTLY ONE POINT

OLIVIER HAUTION

ABSTRACT. We prove an algebraic version of classical theorem in topology, asserting that an abelian p -group action on a smooth projective variety of positive dimension cannot fix exactly one point. In the case of involutions, the number of fixed points cannot be odd. The main tool is a construction originally used by Rost in the context of the degree formula. The framework of diagonalisable p -groups allows us to include the case of base fields of characteristic p .

INTRODUCTION

The following result in algebraic topology was proved in the sixties:

Theorem (Conner-Floyd/Atiyah-Bott). *An orientation preserving diffeomorphism of odd prime power order on a closed oriented manifold of positive dimension cannot have exactly one fixed point.*

This was initially a conjecture of Conner-Floyd [CF64, §45], proved first by Atiyah-Bott [AB68, Theorem 7.1] as a consequence of their fixed point formula, and later reproved by Conner-Floyd [CF66, (8.3)] using equivariant bordism. In the eighties, this result was generalised to actions of abelian p -groups (as opposed to cyclic p -groups) independently by Browder [Bro87, Corollary (1.6)] and Ewing-Stong [ES86, §3].

In the present paper, we discuss the situation in algebraic geometry, and in particular prove the following statement.

Theorem. *Let G be a finite abelian p -group acting on a smooth projective variety X over an algebraically closed field of characteristic unequal to p . If X has no zero-dimensional component, then G cannot fix precisely one point.*

Our main tool is a construction originally due to Rost. When the group \mathbb{Z}/p (with p a prime number) acts on a variety X over a field containing a root of unity of order p (in particular of characteristic unequal to p), Rost defines in [Ros01] a cycle class $\varrho(X)$ in the modulo p Chow group of the fixed locus. He proved the so-called degree formula [Mer01, Theorem 4.1] using this class in the

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case of the action by cyclic permutations of factors on the p -th power $X = Y^p$ of a variety Y , where the fixed locus is the diagonal Y . But this construction happens to be also useful in another extreme case, namely when the fixed locus is finite. In this situation, the degree of the cycle class $\varrho(X)$ is the number of fixed points (modulo p), counted with appropriate multiplicities. A crucial observation for the proof of the theorem above concerns an equivariance property of Rost's construction, when the \mathbb{Z}/p -action extends to the action of a larger abelian group G (we are however unable to construct a functorial lifting of the Rost operation to G -equivariant modulo p Chow groups).

Let us emphasise the difference of context with the degree formula of [Mer01, Theorem 4.1]. The latter is used for questions of arithmetic nature (it asserts the existence of a zero-cycle of a certain degree), a typical consequence being the isotropy of a certain quadratic form. By contrast the theorem above is essentially geometric, and indeed most statements of this paper are not significantly weakened by assuming that the base field is algebraically closed.

Although the consideration actions of ordinary groups — as opposed to algebraic groups — on varieties would suffice to prove the theorem stated above, we chose to write the paper using the more sophisticated framework of diagonalisable groups. This allows in particular to treat the case of infinitesimal diagonalisable p -groups (such as μ_p in characteristic p), which may be of some interest. More importantly, we found that most proofs become more transparent and considerably shorter when the language of diagonalisable groups is used (a notable exception is §5, since the notion of equivariant cycles seems to be more complicated for non-constant groups). We prove the following more general theorem.

Theorem. *Let G be a finite diagonalisable p -group acting on a projective variety X over an arbitrary field. Assume that X has no zero-dimensional connected component. Then the set underlying the fixed locus X^G cannot be a single regular closed point of X of degree prime to p .*

The structure of the paper is as follows. The purpose of the first section is mostly to provide some motivation to the reader. We state the main result in the case of the action of an ordinary p -group (postponing its proof until the end of the paper), and provide a series of examples aimed at testing the sharpness of the statement, and at motivating the consideration of diagonalisable groups in the rest of the paper (see e.g. (1.1.5)). The case of the action of the group $G = \mathbb{Z}/2$ (that is, of involutions) may be dealt with using the construction of [Hau16], and in fact we are able to prove a stronger statement (§1.2): the number of fixed points of an involution may not be odd. In addition, we prove that on a smooth projective variety whose topological Euler characteristic is odd, every involution must fix infinitely many points.

After explaining our notation in §2, we recall in §3 general facts concerning finite diagonalisable groups and their actions on algebraic varieties. We study in some details μ_n -torsors, defining the objects \mathcal{L} and \mathfrak{s} which plays an important role in the sequel.

We then explain in §4 the construction of Rost's operation, already described in the preprints of Rost [Ros01] and Boisvert [Boi08]. Essentially the only new result in this section is the observation that everything works also in characteristic p , provided that \mathbb{Z}/p is replaced with μ_p . Even though we are interested in an operation at the level of Chow groups, we have to consider higher K -cohomology groups in intermediate steps of the construction. Since there is essentially no additional cost, we construct the operation for an arbitrary cycle module M ; in the applications M will be the modulo p Milnor K -theory.

In §5, we prove that if the μ_p -action on a variety extends to an action of a larger finite diagonalisable group G , then the operation of §4 produces a G -equivariant cycle. The results of §4 and §5 are then used in §6 to prove the main theorem.

1. RESULTS FOR ORDINARY GROUPS

In order to state the main theorem, we will use the following terminology concerning group actions on varieties. A more general, but compatible, terminology will be given in §2.2 and used from there on. Let k be a field. An action of an ordinary group G on a k -variety is a group morphism $G \rightarrow \text{Aut}_k(X)$. For $g \in G$, the closed subscheme X^g of X is defined as the equaliser of the automorphism of X induced by g and the identity of X . The fixed locus X^G is the closed subscheme of X defined as the intersection of X^g for $g \in G$. Its set of k -points $X^G(k)$ coincides with the set of fixed points $X(k)^G$.

1.1. Actions of abelian p -groups.

1.1.1. Theorem. *Let X be a projective k -variety without connected component of dimension zero, and G an ordinary abelian p -group acting on X . For every $g \in G$, assume that k contains a root of unity whose order is the order of g . Then the set underlying X^G cannot be a single regular closed point of X of degree prime to p .*

Proof. We may assume that G is finite by (1.1.2) below. By §3.6, the G -action on X corresponds to the action of some finite diagonalisable p -group. Thus the theorem is a special case of (6.5). \square

1.1.2. Lemma. *Let G be an ordinary group acting on a k -variety X . Then there is a finitely generated subgroup G' of G such that $X^G = X^{G'}$.*

Proof. Assume the contrary. We construct a chain of finitely generated subgroups $G_i \subset G_{i+1}$ of G such that $X^{G_{i+1}} \subsetneq X^{G_i}$. Since X is a noetherian topological space, and the fixed loci are closed, we will obtain a contradiction and thus prove the lemma. Let $G_0 = 1$, and assume G_i constructed for $i \geq 0$. By assumption

$X^G \subsetneq X^{G_i}$. It follows that we may find $g \in G$ with $X^g \cap X^{G_i} \subsetneq X^{G_i}$, and we let $G_{i+1} \subset G$ be the subgroup generated by G_i and g . \square

We now illustrate the necessity of the hypotheses of (1.1.1).

1.1.3. Example (G must be abelian). Assume that the characteristic of k is not 2. The symmetric group \mathfrak{S}_4 on four letters acts on \mathbb{P}_k^3 by permuting the coordinates (x_0, x_1, x_2, x_3) , and thus also on the \mathfrak{S}_4 -invariant hyperplane X defined by $x_0 + x_1 + x_2 + x_3 = 0$. Let G be the 2-Sylow subgroup of \mathfrak{S}_4 generated by

$$(x_0, x_1, x_2, x_3) \mapsto (x_1, x_2, x_3, x_0) \text{ and } (x_0, x_1, x_2, x_3) \mapsto (x_2, x_1, x_0, x_3).$$

The group G is a non-abelian 2-group (it is isomorphic to the dihedral group of order 8). The set underlying X^G is the single rational point $[1 : -1 : 1 : -1]$, and X has no zero-dimensional component, being isomorphic to \mathbb{P}_k^2 .

1.1.4. Example (the characteristic of k cannot be p). Assume that the characteristic of k is 2, and consider the action of the $G = \mathbb{Z}/2$ on $X = \mathbb{P}_k^1$ given by the involution $[x : y] \mapsto [y : x]$. Then the set underlying X^G is the single rational point $[1 : 1]$. Note however that the scheme X^G does not coincide with this closed point, in fact $X^G \simeq \operatorname{Spec} k[t]/(t^2)$. This case will be covered by (1.2.1) below.

1.1.5. Example (k must contain enough roots of unity). Assume that the characteristic of k is not p . The endomorphism $[x_0 : \cdots : x_{p-1}] \mapsto [x_1 : \cdots : x_{p-1} : x_0]$ of \mathbb{P}_k^{p-1} has order p , hence induces an action of the group $G = \mathbb{Z}/p$ on \mathbb{P}_k^{p-1} . The hyperplane X defined by $x_0 + \cdots + x_{p-1} = 0$ is G -invariant, and has no zero-dimensional component when p is odd. Over an algebraic closure \bar{k} of k , the fixed locus $(X^G)_{\bar{k}} = (X_{\bar{k}})^G$ consists of the $(p-1)$ points $[1 : \xi : \xi^2 : \cdots : \xi^{p-1}]$, where ξ runs over the primitive p -th of unity in \bar{k} . It follows in particular that X^G is a single closed point of degree $p-1$ when $k = \mathbb{Q}$.

1.2. Involutions. Example (1.1.5) also shows that X^G may consist of n regular points with n prime to p (namely $n = p-1$) when k is algebraically closed and $G = \mathbb{Z}/p$ with $p \neq 2$. This contrasts with the situation in the case $p = 2$ (see (1.2.2.i) below), where one may also handle singular fixed points and base fields of characteristic two, provided that points are counted with appropriate multiplicities:

1.2.1. Theorem. *Let X be a projective k -variety of pure dimension $d > 0$. Assume that $G = \mathbb{Z}/2$ acts on X with $\dim X^G = 0$. Then the integer*

$$\sum_{x \in X^G} e_x[k(x) : k]$$

is even, where e_x denotes Samuel's multiplicity of the primary ideal $I = \ker(\mathcal{O}_{X,x} \rightarrow \mathcal{O}_{X^G,x})$ of the local ring $A = \mathcal{O}_{X,x}$; in other words for $n \geq 1$ we have

$$\operatorname{length}_A(A/I^n) = e_x \cdot \frac{n^d}{d!} + \beta_n n^{d-1},$$

where β_n has a finite limit as n tends to ∞ (see [Bou06, VIII, §7, N°1, Definition 1] or [Ful98, Example 4.3.4]).

Proof. The cycle class $\mathcal{S}_X \in \mathrm{CH}(X^G)$ of [Hau16, §4.2] is defined as the Segre class of the normal cone of the closed embedding $X^G \rightarrow X$. Since $\dim X^G = 0$, it follows from [Ful98, Example 4.3.4] that the integer of the proposition coincides with the degree of the cycle class \mathcal{S}_X . This degree must be even, by the degree formula [Hau16, Theorem 4.2.4] applied to the proper G -equivariant morphism $X \rightarrow \mathrm{Spec} k$ (which has degree zero since X has no zero-dimensional component). \square

1.2.2. Corollary. *Let k be an algebraically closed field of characteristic unequal to two, and X a projective k -variety without zero-dimensional connected component.*

- (i) *The set of fixed points of a k -involution of X cannot consist in an odd number of regular points.*
- (ii) *If X is smooth, and the Euler characteristic of X relative to the 2-adic cohomology*

$$\chi(X) = \sum_i (-1)^i \dim_{\mathbb{Q}_2} H_{\text{ét}}^i(X, \mathbb{Q}_2)$$

is odd, then any k -involution of X fixes infinitely many points.

Proof. Assume that $G = \mathbb{Z}/2$ acts on X . Let us decompose the set of irreducible components of X into G -orbits X_i . It will suffice to prove the statements after replacing X with each of the X_i (for (i) this follows from the fact that $X_i \cap X_j$ contains no regular point of X when $i \neq j$; in case (ii) we have $X_i \cap X_j = \emptyset$ for $i \neq j$, and therefore $\chi(X) = \sum_i \chi(X_i)$). Thus we may assume that X is equidimensional.

(i): Assume the set underlying X^G is a finite number of regular points. Due to the assumptions on the field k , the scheme X^G is smooth (indeed X^G is contained in a smooth G -invariant open subscheme of X ; then use [Edi92, Proposition 3.4], or alternatively (3.6.1) and (3.3.2) below). Then the result follows from (1.2.1): if $x \in X^G$, the kernel of $\mathcal{O}_{X,x} \rightarrow \mathcal{O}_{X^G,x}$ is the maximal ideal of the regular local ring $\mathcal{O}_{X,x}$, so that $e_x = 1$ by [Bou06, VIII, §7, N°1, Proposition 2].

(ii): We have $\chi(X) = \chi(X^G) \pmod{2}$ by [Ser09, §7.2]. If $\dim X^G \leq 0$, then the integer $\chi(X^G)$ is the number of fixed points, and must be even by (i). \square

1.2.3. Remark. The integer $\chi(X)$ of (1.2.2.ii) coincides with the degree of the top Chern class of the tangent bundle of X (see e.g. [Hau15, Proposition 3.2]).

1.2.4. Example ((1.2.2) does not generalise to other 2-groups). Assume that the characteristic of k is not two. The commuting involutions

$$[x : y : z] \mapsto [-x : y : z] \quad \text{and} \quad [x : y : z] \mapsto [x : -y : z]$$

give an action of $G = \mathbb{Z}/2 \times \mathbb{Z}/2$ on $X = \mathbb{P}_k^2$. Then the set underlying X^G consists of the three points $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]$. In addition, $\chi(X) = 3$ is odd.

For an example with G cyclic and the same X , assume that k contains a primitive fourth root of unity i . The automorphism of order four

$$[x : y : z] \mapsto [-x : iy : z]$$

induces an action of $G = \mathbb{Z}/4$ on X . The set underlying X^G again consists of the three points $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]$.

2. NOTATION AND BASIC FACTS

2.1. Varieties. The letter k will denote a base field. A variety, or k -variety, will be a quasi-projective scheme over k . The residue field at a point x of a k -variety will be denoted by $k(x)$. When $X = \operatorname{Spec} A$ is an affine variety and $f \in A$, we will denote by $D(f)$ the principal open subscheme $\operatorname{Spec} A[f^{-1}]$ of X .

A point x of a variety X will be called regular if the local ring $\mathcal{O}_{X,x}$ is regular, and we will say that a variety is regular when all its points are regular. A morphism of varieties is a morphism of k -schemes. A flat morphism of varieties will always mean a flat morphism with a relative dimension. When X is covered by open subschemes U_i for $i \in I$, the words “replacing X with the cover $\{U_i, i \in I\}$ ” will mean successively replacing X with each U_i .

2.2. Algebraic groups. We refer e.g. to [MFK94, Chapter 0, §1] for the basic definitions concerning algebraic groups and their actions on varieties. An algebraic group will mean a group scheme of finite type over k , and actions will always be over k .

If an algebraic group G acts on X , an open or closed subscheme Y of X will be called G -invariant if the restriction of the action morphism $G \times Y \rightarrow X$ factors through Y . The scheme-theoretic closure of a G -invariant open subscheme is a G -invariant closed subscheme.

When an algebraic group G is finite, we denote by $|G|$ its order, defined as the dimension of the k -vector space $H^0(G, \mathcal{O}_G)$. A finite algebraic group will be called a p -group if its order is a power of p .

2.3. Algebraic Cycles. We will use the notation of [Ful98] concerning algebraic cycles, with the following variations. The group of cycles on a variety X , denoted by $\mathcal{Z}(X)$ (instead of $Z_*(X)$), is the free abelian group generated by the classes $[Z]$, for Z an integral closed subscheme of X . The Chow group of X will be denoted by $\operatorname{CH}(X)$ (instead of $A_*(X)$); this is the quotient of $\mathcal{Z}(X)$ modulo the relation of rational equivalence.

2.4. The degree of a morphism. We say that a morphism $f: Y \rightarrow X$ has degree m if $f_*[Y] = m \cdot [X] \in \mathcal{Z}(X)$. We will also write $m = \deg f$. The morphism f always has a degree when X is integral and Y has no irreducible component of dimension $< \dim X$.

Setting $\deg[Z] = \deg(Z \rightarrow \operatorname{Spec} k)$ for Z an integral closed subscheme of X induces a morphism $\mathcal{Z}(X) \rightarrow \mathbb{Z}$, which descends to $\deg: \operatorname{CH}(X) \rightarrow \mathbb{Z}$ when X

is complete. When p is a prime, we will also write \deg for induced morphism $\mathrm{CH}(X)/p \rightarrow \mathbb{F}_p$.

2.5. The sheaf of invertible functions. We denote by \mathbb{G}_m the Zariski sheaf of groups such that $H^0(X, \mathbb{G}_m) = H^0(X, \mathcal{O}_X)^\times$ for any variety X . If $t \in H^0(X, \mathbb{G}_m)$ and $f: Y \rightarrow X$, we will often write $t \in H^0(Y, \mathbb{G}_m)$ instead of $f^*(t)$.

2.6. Cycle modules. A cycle module will mean a cycle module over $\mathrm{Spec} k$, in the sense of [Ros96]. When M is a cycle module and X a variety, the complex of cycles on X with coefficients in M will be denoted by $C(X, M)$, and the Chow group with coefficients in M (the direct sum of the homology groups of $C(X, M)$) by $A(X, M)$. The differential will be denoted by $d: C(X, M) \rightarrow C(X, M)$. When E is a (closed or open) subscheme of X we denote by $x \mapsto x|_E$ the projection $C(X, M) \rightarrow C(E, M)$. An element $t \in H^0(X, \mathbb{G}_m)$ induces an endomorphism $\{t\}$ of $C(X, M)$ and of $A(X, M)$, see [Ros96, (3.6), (4.6.3)].

We will use the following statement which does not appear explicitly in [Ros96].

2.6.1. Lemma. *Let $Y \rightarrow X$ and $i: Z \rightarrow Y$ be two closed embeddings. Denote by $u: X - Z \rightarrow X - Y$ the open immersion. Then for any cycle module M the following diagram commutes.*

$$\begin{array}{ccc} A(X - Z, M) & \xrightarrow{\delta_Z} & A(Z, M) \\ u^* \downarrow & & \downarrow i_* \\ A(X - Y, M) & \xrightarrow{\delta_Y} & A(Y, M) \end{array}$$

Proof. When B is a locally closed subscheme of X , we will view $C(B, M)$ as a subgroup of $C(X, M)$. Any element of $A(X - Z, M)$ is represented by some $\alpha \in C(X - Z, M) \subset C(X, M)$ such that $d(\alpha)|_{X-Z} = 0$. Writing $\beta = \alpha|_{X-Y}$ and $\gamma = \alpha|_{Y-Z}$, we have $\alpha = \beta + \gamma$. Now we compute in $C(Y, M)$:

$$\begin{aligned} \delta_Y \circ u^*(\alpha) &= d(\beta)|_Y \\ &= d(\beta)|_Z + d(\beta)|_{Y-Z} \\ &= d(\alpha)|_Z - d(\gamma)|_Z + d(\beta)|_{Y-Z} \\ &= d(\alpha)|_Z - d(\gamma)|_Z - d(\gamma)|_{Y-Z} && \text{since } d(\alpha) \in C(Z, M) \\ &= d(\alpha)|_Z - d(\gamma)|_Y \\ &= i_* \circ \delta_Z(\alpha) - d(\gamma)|_Y. \end{aligned}$$

Since $\gamma \in C(Y - Z, M) \subset C(Y, M)$ and Y is closed in X , the element $d(\gamma)|_Y$ belongs to the image of the differential of $C(Y, M)$, hence vanishes in $A(Y, M)$. The statement follows. \square

2.7. The first Chern class. Let M be a cycle module. Let $q: L \rightarrow X$ be a line bundle with zero section $i: X \rightarrow L$ and \mathcal{O}_X -module of sections \mathcal{L} . Its first Chern class operator is

$$c_1(\mathcal{L}) = c_1(L) = (q^*)^{-1} \circ i_*: A(X, M) \rightarrow A(X, M),$$

and we will also consider the operator

$$c(-\mathcal{L}) = \sum_{j=0}^{\dim X} (-c_1(\mathcal{L}))^j: A(X, M) \rightarrow A(X, M).$$

2.7.1. Lemma. *Let L be a line bundle on a variety X , and M a cycle module.*

(i) *Let $t \in H^0(X, \mathbb{G}_m)$. Then*

$$c_1(L) \circ \{t\} = \{t\} \circ c_1(L): A(X, M) \rightarrow A(X, M).$$

(ii) *Let $f: Y \rightarrow X$ be a flat morphism. Then*

$$f^* \circ c_1(L) = c_1(f^*L) \circ f^*: A(X, M) \rightarrow A(Y, M).$$

(iii) *Let $f: Y \rightarrow X$ be a proper morphism. Then*

$$f_* \circ c_1(f^*L) = c_1(L) \circ f_*: A(Y, M) \rightarrow A(X, M).$$

(iv) *Let Y be a closed subscheme of X , and $\delta: A(X - Y, M) \rightarrow A(Y, M)$ the connecting homomorphism. Then*

$$\delta \circ c_1(L|_{X-Y}) = c_1(L|_Y) \circ \delta: A(X - Y, M) \rightarrow A(Y, M).$$

Proof. This follows from [Ros96, (4.1), (4.2.1), (4.3.1), (4.4)]. \square

The following direct construction of the first Chern class will be used in §5.

2.7.2. Lemma. *Let \mathcal{L} be an invertible \mathcal{O}_X -module. Let U be a dense open subscheme of X and $l \in H^0(U, \mathcal{L})$ a nowhere vanishing section. Let U_i be a covering of X by open subschemes, and $l_i \in H^0(U_i, \mathcal{L})$ nowhere vanishing sections. Let $s_i \in H^0(U \cap U_i, \mathbb{G}_m)$ be such that $l_i s_i = l$ on $U \cap U_i$. Then the cycle class $c_1(\mathcal{L})[X] \in \text{CH}(X)$ is represented by a cycle $z \in \mathcal{Z}(X)$ such that $z|_{U_i}$ is the image of $[U \cap U_i]$ under the composite (we denote by $u_i: U \cap U_i \rightarrow U_i$ the open immersion)*

$$\mathcal{Z}(U \cap U_i) \xrightarrow{\{s_i\}} C(U \cap U_i, K_1) \xrightarrow{(u_i)_*} C(U_i, K_1) \xrightarrow{d} \mathcal{Z}(U_i).$$

Proof. We may assume that X is integral. Then for each i such that $U_i \neq \emptyset$, we may view each s_i as an invertible rational function on X , and the family (U_i, s_i) defines a Cartier divisor S on X such that $\mathcal{O}(S) \simeq \mathcal{L}$. By [EKM08, 57.24] we may take for z the Weil divisor associated with S . \square

2.8. The sheaf \mathbb{G}_m/n . The letter n will denote an integer ≥ 1 .

2.8.1. Definition. We denote by \mathbb{G}_m/n the cokernel of the n -th power map $\mathbb{G}_m \rightarrow \mathbb{G}_m$ in the category of Zariski sheaves.

2.8.2. Remark. The sheaf \mathbb{G}_m/n is the Zariski sheafification of the presheaf $U \mapsto H_{fppf}^1(U, \mu_n)$.

Let X be a variety and M a cycle module such that $n \cdot M = 0$. Let $t \in H^0(X, \mathbb{G}_m/n)$. The stalk of t at a point $x \in X$ is an element of $(\mathcal{O}_{X,x})^\times / (\mathcal{O}_{X,x})^{\times n}$, and its image in $k(x)^\times / k(x)^{\times n}$ acts on the left $k(x)^\times$ -module $M(k(x))$ (which is assumed to be n -torsion). Thus we obtain a group morphism

$$\{t\}: C(X, M) \rightarrow C(X, M).$$

Note that any point of X is contained in an open subscheme U such that (the restriction of) t lifts to a section $t' \in H^0(U, \mathbb{G}_m)$. Then the endomorphisms $\{t\}$ and $\{t'\}$ of $C(U, M)$ coincide.

2.8.3. Proposition. *Let X be a variety and M a cycle module such that $n \cdot M = 0$. Let $t \in H^0(X, \mathbb{G}_m/n)$. Then (at the cycle level $C(-, M)$):*

- (i) $d \circ \{t\} = -\{t\} \circ d$, where $d: C(X, M) \rightarrow C(X, M)$ is the differential.
- (ii) If $f: Y \rightarrow X$ is a morphism, then $f_* \circ \{f^*t\} = \{t\} \circ f_*$.
- (iii) If $f: Y \rightarrow X$ is a flat morphism, then $f^* \circ \{t\} = \{f^*t\} \circ f^*$.
- (iv) For any $u \in H^0(X, \mathbb{G}_m)$, we have $\{u\} \circ \{t\} = -\{t\} \circ \{u\}$.

Proof. Each of the statement may be proved after replacing X with an open cover, so that we may assume that t lifts to $H^0(X, \mathbb{G}_m)$. Thus the statements follow respectively from [Ros96, (4.6.3), (4.2.1), (4.3.1)], and the graded-commutativity of Milnor K -theory. \square

2.9. Gradings. We now state our terminology concerning graded algebras and modules. These notions extend in an obvious way to sheaves of algebras and modules over them.

Let H be an (ordinary) abelian group and A a commutative ring with unity. A H -grading on a commutative A -algebra R is a decomposition of the A -module $R = \bigoplus_{h \in H} R_h$ such that $R_h \cdot R_i \subset R_{h+i}$ for all $h, i \in H$. A H -grading on an R -module M is a decomposition of the A -module $M = \bigoplus_{h \in H} M_h$ such that $R_h \cdot M_i \subset M_{h+i}$ for all $h, i \in H$. Elements of $\bigcup_{h \in H} M_h$ are called homogeneous. A morphism of H -graded modules is a morphism compatible with the gradings.

A graded submodule of a H -graded module M is a submodule $N \subset M$ which is generated by homogeneous elements. It is equivalent to require that for every $n \in N$ and $h \in H$ the component of n in M_h belong to N . In this case, the modules N and M/N inherit H -gradings such that $N_h = N \cap M_h$ and $(M/N)_h = M_h/N_h$.

A graded ideal of a H -graded A -algebra R is a graded submodule of R . If I, J are graded ideals, then the ideal IJ is graded.

3. ACTIONS OF FINITE DIAGONALISABLE GROUPS ON VARIETIES

3.1. Quotients by finite algebraic groups. In §3.1 and §3.2, we gather some results on finite algebraic group actions which will repeatedly be used in the paper without explicit reference.

3.1.1. Proposition. *Let G be a finite algebraic group acting on a variety X .*

- (i) *The variety X is covered by affine G -invariant open subschemes.*
- (ii) *Any open subscheme U of X contains a G -invariant open subscheme U' such that:*
 - (a) *If U is closed in X , the same is true for U' .*
 - (b) *Any G -invariant closed subscheme of X contained in U is also contained in U' .*
 - (c) $\dim(X - U) = \dim(X - U')$.

Proof. (i): See [SGA III₁, V, §5], the main point is that our varieties are quasi-projective, so that any finite subset of X is contained in an affine open subscheme.

(ii): Denote by $p: G \times X \rightarrow X$ the second projection, and by $a: G \times X \rightarrow X$ the action morphism. Consider the automorphism $\varepsilon: G \times X \rightarrow G \times X$ given by $(g, x) \mapsto (g^{-1}, g \cdot x)$. Since p is open (being flat of finite type) and finite, the same is true for $a = p \circ \varepsilon$. Let F be the set-theoretic closed complement of U in X , and $S = a(p^{-1}F)$. Denoting by $\mu_G: G \times G \rightarrow G$ the group operation, we have

$$a(p^{-1}S) = a \circ (\text{id}_G \times a)(G \times G \times F) = a \circ (\mu_G \times \text{id}_F)(G \times G \times F) = a(G \times F) = S.$$

This proves that $U' = X - S$ is G -invariant. If F is open in X , then so is S (because a is open), proving (a). If Z is a G -invariant closed subscheme of X such that $Z \cap F = \emptyset$, then the set $a^{-1}Z = p^{-1}Z$ does not meet $p^{-1}F$ in $G \times X$, so that $S \cap Z = \emptyset$, proving (b). Finally we have $\dim S = \dim F$ (because p and a are finite, see [EGA IV₂, (5.4.2)]), proving (c). \square

3.1.2. Proposition. *Let G be a finite algebraic group acting on a variety X . Then there is a categorical quotient $\varphi = \varphi_X: X \rightarrow X/G$ in the category of k -varieties. In addition:*

- (i) *The morphism φ is finite and surjective.*
- (ii) *If U is a G -invariant open subscheme of X , then U/G is an open subscheme of X/G , and $U = \varphi_X^{-1}(U/G)$.*

Proof. See [SGA III₁, V, §5] for the existence of the quotient, and for (i). Assertion (ii) follows from [SGA III₁, V, Lemme 1.1]. \square

When $f: Y \rightarrow X$ is a G -equivariant morphism, we will denote by $f/G: Y/G \rightarrow X/G$ the induced morphism.

3.2. Finite diagonalisable groups. (see [Wat79, §2.2] or [SGA III₁, I, §4.4]) Let Γ be a finite abelian (ordinary) group. The functor associating to each commutative k -algebra R the group of group morphisms $\Gamma \rightarrow R^\times$ is represented by a

finite commutative algebraic group $D(\Gamma)$. Algebraic groups of this type are called *finite diagonalisable*. The coordinate ring of $D(\Gamma)$ is the group algebra $k[\Gamma]$ over k , defined as the k -vector space on the basis e_g for $g \in \Gamma$, with multiplication given by $e_g \cdot e_h = e_{g+h}$, comultiplication by $e_g \mapsto e_g \otimes e_g$, coinverse by $e_g \mapsto e_{-g}$ and counit by $e_g \mapsto 1$.

We denote by \widehat{G} the character group of an algebraic group G ; this is the (ordinary) abelian group of group-like elements in $H^0(G, \mathcal{O}_G)$. The associations $G \mapsto \widehat{G}$ and $\Gamma \mapsto D(\Gamma)$ induce an anti-equivalence between the category of finite abelian groups and the category of finite diagonalisable algebraic groups. For every integer $n \geq 1$, we let $\mu_n = D(\mathbb{Z}/n)$ be the finite diagonalisable group such that $\widehat{\mu_n} = \mathbb{Z}/n$.

In the affine case, actions of a finite diagonalisable groups correspond to gradings of the coordinate ring:

3.2.1. Lemma ([SGA III₁, I, 4.7.3.1]). *Let G be a finite diagonalisable group and $\psi: X \rightarrow S$ an affine morphism of varieties. The datum of a G -action on X over S is equivalent to that of a \widehat{G} -grading on the \mathcal{O}_S -algebra $\psi_*\mathcal{O}_X$.*

3.2.2. Proposition. *Let G be a finite diagonalisable group acting on a variety X . Then the morphism $\mathcal{O}_{X/G} \rightarrow \varphi_*\mathcal{O}_X$ induces an isomorphism $\mathcal{O}_{X/G} \simeq (\varphi_*\mathcal{O}_X)_0$.*

Proof. By [SGA III₁, I, §4.7.3], the action morphism $G \times X \rightarrow X$ corresponds to the morphism of $\mathcal{O}_{X/G}$ -algebras $\varphi_*\mathcal{O}_X \rightarrow \varphi_*\mathcal{O}_X \otimes_k k[\widehat{G}]$ given by $\sum_{g \in \widehat{G}} \pi_g \otimes e_g$, where $\pi_g: \varphi_*\mathcal{O}_X \rightarrow \varphi_*\mathcal{O}_X$ is the projection onto the g -th component. Thus [SGA III₁, Théorème 4.1 (i)] implies that $\mathcal{O}_{X/G} \rightarrow \varphi_*\mathcal{O}_X$ is the equaliser in the category of sheaves of rings on X/G of the two morphisms $\varphi_*\mathcal{O}_X \rightarrow \varphi_*\mathcal{O}_X \otimes_k k[\widehat{G}]$ given by $\text{id} \otimes 1$ and $\sum_{g \in \widehat{G}} \pi_g \otimes e_g$. Thus $\mathcal{O}_{X/G}$ is the intersection of the kernels of π_g for $g \in \widehat{G} - \{0\}$, which coincides with $(\varphi_*\mathcal{O}_X)_0$. \square

3.2.3. Lemma. *Let G be a finite diagonalisable group and $f: Y \rightarrow X$ a G -equivariant morphism. If f is respectively*

- (i) *proper,*
- (ii) *a closed embedding,*

then the same is true for f/G .

Proof. (i): The morphism f/G is proper because φ_X is proper and φ_Y is surjective [EGA II, (5.4.2.ii) and (5.4.3.ii)].

(ii): We may assume that f is given by a surjective morphism of \widehat{G} -graded k -algebras $A \rightarrow B$. Then $A_0 \rightarrow B_0$ is surjective. \square

3.2.4. Definition. Let X be a variety with an action of a finite diagonalisable group G . A G -equivariant structure on a quasi-coherent \mathcal{O}_X -module \mathcal{F} is a \widehat{G} -grading on the $\varphi_*\mathcal{O}_X$ -module $\varphi_*\mathcal{F}$. A G -representation over k (the case $X = \text{Spec } k$) is a k -vector space \mathcal{V} together with a k -linear decomposition $\mathcal{V} = \bigoplus_{g \in \widehat{G}} \mathcal{V}_g$. The dual representation \mathcal{V}^\vee is defined by $(\mathcal{V}^\vee)_g = (\mathcal{V}_{-g})^\vee \subset \mathcal{V}^\vee$ for any $g \in \widehat{G}$.

If $f: Y \rightarrow X$ is a G -equivariant morphism and \mathcal{F} a quasi-coherent G -equivariant \mathcal{O}_X -module, the morphism of $\mathcal{O}_{Y/G}$ -modules $\varphi_{Y*} \circ f^* \mathcal{F} \rightarrow (f/G)^* \circ \varphi_{X*} \mathcal{F}$ is an isomorphism [EGA II, (1.5.2)]. This induces a G -equivariant structure on the \mathcal{O}_Y -module $f^* \mathcal{F}$.

3.2.5. Definition. Let G be finite diagonalisable group acting on a variety X . Let \mathcal{F} be a G -equivariant \mathcal{O}_X -module and $g \in \widehat{G}$. We say that a section in $H^0(X, \mathcal{F})$ has weight g if it belongs to the subgroup

$$H^0(X/G, (\varphi_* \mathcal{F})_g) \subset H^0(X/G, \varphi_* \mathcal{F}) = H^0(X, \mathcal{F}).$$

3.3. The fixed locus.

3.3.1. Definition. Let G be a finite diagonalisable group acting on a variety X . Since the quotient morphism $\varphi: X \rightarrow X/G$ is affine, there is a unique coherent ideal \mathcal{I} of \mathcal{O}_X such that the ideal $\varphi_* \mathcal{I}$ of $\varphi_* \mathcal{O}_X$ is generated by $(\varphi_* \mathcal{O}_X)_g$ for $g \in \widehat{G} - \{0\}$. We let X^G be the closed subscheme of X defined by the ideal \mathcal{I} . The functor associating to a variety T with trivial G -action the set of G -equivariant morphisms $T \rightarrow X$ is represented by the variety X^G .

Note that X^G is a G -invariant closed subscheme of X with trivial G -action. A G -equivariant morphism $f: Y \rightarrow X$ induces a morphism $f^G: Y^G \rightarrow X^G$. When Y and X are two varieties with a G -action, we will endow the product $X \times Y$ (over k) with the diagonal G -action; then $(X \times Y)^G = X^G \times Y^G$.

3.3.2. Lemma. *Let G be a finite diagonalisable group acting on a regular variety X . Then the variety X^G is regular.*

Proof. Let x be a point of X^G and $R = \mathcal{O}_{X,x}$. Any open neighborhood of x contains a G -invariant open neighborhood by (3.1.1.ii.b), which in addition may be taken to be affine by (3.1.1.i). Therefore the k -algebra R is a direct limit of \widehat{G} -graded k -algebras, and is thus naturally \widehat{G} -graded [Bou70, II, §11, N°3, Remarque 3)]. In addition it follows from [Bou70, II, §6, N°2, Propositions 3 and 4] that the ideal I of R generated by R_g for $g \in \widehat{G} - \{0\}$ is the kernel of $R \rightarrow \mathcal{O}_{X^G,x}$. It will suffice to prove that the noetherian local ring R/I is regular.

The maximal ideal \mathfrak{m} of R is \widehat{G} -graded, as is any ideal containing I (if $m \in \mathfrak{m}$, denoting by m_g its component in R_g , we have $m_g \in I \subset \mathfrak{m}$ if $g \neq 0$, and therefore also $m_0 = m - \sum_{g \neq 0} m_g \in \mathfrak{m}$). It follows that the ideal \mathfrak{m}^2 is \widehat{G} -graded (being the product of two graded ideals), and is thus a graded submodule of \mathfrak{m} . Let $\kappa = R/\mathfrak{m} = R_0/\mathfrak{m}_0$. The κ -vector space $V = \mathfrak{m}/\mathfrak{m}^2$ splits as $\bigoplus_{g \in \widehat{G}} V_g$, where $V_g = \mathfrak{m}_g/(\mathfrak{m}^2)_g$. Lifting a κ -basis of each V_g to \mathfrak{m}_g , we obtain a regular system of parameters x_1, \dots, x_n of R and elements $g_1, \dots, g_n \in \widehat{G}$ such that $x_i \in R_{g_i}$ for each i . Let J be the ideal of R generated by those x_i such that $g_i \neq 0$. To conclude the proof, it will suffice to prove that $I = J$. Clearly $J \subset I$. Since $\mathfrak{m} = x_1 R + \dots + x_n R$, it follows that $R_g = \mathfrak{m}_g = x_1 R_{g-g_1} + \dots + x_n R_{g-g_n}$ for each

$g \neq 0$. But for such g , we have $x_i R_{g-g_i} \subset J$ if $g_i \neq 0$, and $x_i R_{g-g_i} \subset \mathfrak{m}I$ if $g_i = 0$. Thus $I \subset \mathfrak{m}I + J$, which by Nakayama's lemma implies that $I = J$. \square

3.3.3. Lemma. *Let G be a finite diagonalisable group acting on a variety X . Let N be the normal cone of the closed embedding $X^G \rightarrow X$, endowed with its natural G -action. Then the closed subscheme N^G of N coincides with the zero-section X^G .*

Proof. We may assume that X is the spectrum of a \widehat{G} -graded k -algebra A . The closed subscheme X^G of X is defined by the ideal $I \subset A$ generated by A_g for $g \in \widehat{G} - \{0\}$. Since I is a graded ideal of A , the associated graded ring $S = \bigoplus_{n \geq 0} I^n / I^{n+1}$ is a $\mathbb{Z} \times \widehat{G}$ -graded k -algebra. The closed subscheme N^G of N is defined by the ideal $P \subset S$ generated by $S_{n,g}$ for $n \in \mathbb{Z}$ and $g \in \widehat{G} - \{0\}$. Then for $g \in \widehat{G} - \{0\}$, we have $I_g = A_g$ and thus $S_{0,g} = 0$. It follows that $P_{0,g} = 0$ for any $g \in \widehat{G}$.

The ideal I_0 of A_0 is generated by the products $A_g \cdot A_{-g}$ for $g \in \widehat{G} - \{0\}$. Thus $I_0 \subset I^2$, and $S_{1,0} = I_0 / (I^2)_0 = 0$. It follows that $P_{1,g} = S_{1,g}$ for any $g \in \widehat{G}$. Since any element of non-zero \mathbb{Z} -degree in S is a product of elements of \mathbb{Z} -degree one, it follows that $P_{n,g} = S_{n,g}$ for any $n \in \mathbb{Z} - \{0\}$ and $g \in \widehat{G}$. This proves that the map $A/I \rightarrow S/P$ is bijective, as required. \square

3.4. Free actions.

3.4.1. Definition. Let G be a finite diagonalisable group acting on a variety X . We say that G acts freely on X if for every $g, h \in \widehat{G}$ the morphism

$$(\varphi_* \mathcal{O}_X)_g \otimes_{\mathcal{O}_{X/G}} (\varphi_* \mathcal{O}_X)_h \rightarrow (\varphi_* \mathcal{O}_X)_{g+h}$$

is an isomorphism ($\varphi: X \rightarrow X/G$ denotes the quotient morphism, see (3.1.2)).

Note that if the finite diagonalisable G acts freely on X , then the $\mathcal{O}_{X/G}$ -modules $(\varphi_* \mathcal{O}_X)_g$ are invertible, and thus the morphism $\varphi: X \rightarrow X/G$ is flat and finite of degree $|G|$. In addition, it follows from (3.4.2.iii) below (applied to the action morphism $G \times X \rightarrow X$) that $X \rightarrow X/G$ is a G -torsor in the fppf topology.

3.4.2. Lemma. *Let G be a finite diagonalisable group and $f: Y \rightarrow X$ a G -equivariant morphism. Assume that G acts freely on X . Then:*

- (i) *The group G acts freely on Y .*
- (ii) *For every $g \in \widehat{G}$ the morphism $(f/G)^*(\varphi_{X*} \mathcal{O}_X)_g \rightarrow (\varphi_{Y*} \mathcal{O}_Y)_g$ is an isomorphism.*
- (iii) *The following square is cartesian:*

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y/G & \longrightarrow & X/G \end{array}$$

Proof. We may assume that f is given by a morphism of \widehat{G} -graded k -algebras $A \rightarrow B$, and moreover that for every $g \in \widehat{G}$ the A_0 -module A_g is free with basis a_g . Then each a_g is invertible in A , and its image $b_g \in B_g$ is invertible in B . Then the map $x \mapsto b_g \otimes (b_g^{-1}x)$ is an inverse to the morphism $B_g \otimes_{B_0} B_h \rightarrow B_{g+h}$, proving (i). To prove (ii), observe that the morphism $A_g \otimes_{A_0} B_0 \rightarrow B_g$ is bijective (an inverse is given by $x \mapsto a_g \otimes (b_g^{-1}x)$). Taking the direct sum over $g \in \widehat{G}$, it follows that $A \otimes_{A_0} B_0 \rightarrow B$ is an isomorphism, proving (iii). \square

3.4.3. Lemma. *If a finite diagonalisable group G acts freely on X , any quasi-coherent \mathcal{O}_X -module admitting a G -equivariant structure (see (3.2.4)) is the pull-back of a quasi-coherent $\mathcal{O}_{X/G}$ -module.*

Proof. Let \mathcal{F} be a G -equivariant quasi-coherent \mathcal{O}_X -module, and \mathcal{G} the quasi-coherent $\mathcal{O}_{X/G}$ -module $(\varphi_*\mathcal{F})_0$. To prove that the natural G -equivariant morphism $\alpha: \varphi^*\mathcal{G} \rightarrow \varphi^*\varphi_*\mathcal{F} \rightarrow \mathcal{F}$ is an isomorphism, we may assume that X is the spectrum of \widehat{G} -graded k -algebra A , and that for each $g \in \widehat{G}$ the A_0 -module A_g is free with basis a_g . Then \mathcal{F} corresponds to a graded A -module F , and \mathcal{G} corresponds to F_0 . For $g \in \widehat{G}$ the g -th component of α corresponds to the morphism $F_0 \otimes_{A_0} A_g \rightarrow F_g$. An inverse of the latter is given by $f \mapsto (a_g^{-1}f) \otimes a_g$. \square

3.5. μ_n -torsors.

3.5.1. Definition. Let X be a variety with a free μ_n -action (see (3.4.1)). We denote by \mathcal{L}_X the invertible \mathcal{O}_{X/μ_n} -module $(\varphi_*\mathcal{O}_X)_1$, where 1 denotes the canonical generator of $\widehat{\mu_n} = \mathbb{Z}/n$. If $f: Y \rightarrow X$ is a μ_n -equivariant morphism, then $(f/\mu_n)^*(\mathcal{L}_X) \simeq \mathcal{L}_Y$ by (3.4.2.ii), so that we will usually write \mathcal{L} instead of \mathcal{L}_X .

3.5.2. Definition. Let X be a variety with a free μ_n -action (see (3.4.1)). We construct below a section \mathfrak{s}_X over X/μ_n of the sheaf \mathbb{G}_m/n defined in (2.8.1). If U is a μ_n -invariant open subscheme of X such that $\mathcal{L}_U = \mathcal{L}_X|_{U/\mu_n}$ is free with basis a , the image of $a^{\otimes n}$ under the morphism $(\mathcal{L}_U)^{\otimes n} \rightarrow \mathcal{O}_{U/\mu_n}$ is an element of $H^0(U/\mu_n, \mathbb{G}_m)$, whose image in $H^0(U/\mu_n, \mathbb{G}_m/n)$ we denote by \mathfrak{s}_U . The element \mathfrak{s}_U does not depend on the choice of a , hence this construction glue to define a section $\mathfrak{s}_X \in H^0(X/\mu_n, \mathbb{G}_m/n)$.

If $f: Y \rightarrow X$ is a μ_n -equivariant morphism, then $(f/\mu_n)^*(\mathfrak{s}_X) = \mathfrak{s}_Y$, so that we will usually write \mathfrak{s} instead of \mathfrak{s}_X .

3.5.3. Remark. Assume that μ_n acts freely on a variety X .

- (i) By (2.8.2), we have a morphism $H_{fppf}^1(X/\mu_n, \mu_n) \rightarrow H^0(X/\mu_n, \mathbb{G}_m/n)$, and $\mathfrak{s}_X \in H^0(X/\mu_n, \mathbb{G}_m/n)$ is the image of the class of the μ_n -torsor $X \rightarrow X/\mu_n$.
- (ii) The class of the invertible \mathcal{O}_{X/μ_n} -module \mathcal{L}_X is the image of the class of the μ_n -torsor $X \rightarrow X/\mu_n$ under the morphism $H_{fppf}^1(X/\mu_n, \mu_n) \rightarrow H_{fppf}^1(X/\mu_n, \mathbb{G}_m) = \text{Pic}(X)$ induced by the Kummer sequence $1 \rightarrow \mu_n \rightarrow \mathbb{G}_m \rightarrow \mathbb{G}_m \rightarrow 1$.

3.5.4. Example. Let $R = k[t]$ be the coordinate ring of \mathbb{A}^1 . We define a \mathbb{Z}/n -grading on the k -algebra R by letting R_i , for $i \in \mathbb{Z}/n$, be the set of polynomials in $k[t]$ whose t^j -th coefficient vanishes unless $i = j \pmod n$. This gives μ_n -action on \mathbb{A}^1 . The fixed locus $(\mathbb{A}^1)^{\mu_n}$ is the closed point 0. The action is free on the open subscheme $X = \mathbb{A}^1 - 0$, and the quotient X/μ_n is the spectrum of $S = k[t^n, t^{-n}]$. The element t is a basis of the S -module S_1 , so that the invertible \mathcal{O}_{X/μ_n} -module \mathcal{L}_X is trivial, and $s \in H^0(X/\mu_n, \mathbb{G}_m/n)$ is the image of $t^n \in H^0(X/\mu_n, \mathbb{G}_m)$.

3.5.5. Lemma. *Let $r \in \mathbb{Z}$, and $D \rightarrow X$ be a μ_n -equivariant principal effective Cartier divisor, given by a section of \mathcal{O}_X of weight $r \pmod n$ in $\mathbb{Z}/n = \widehat{\mu_n}$ (see (3.2.5)). If μ_n acts freely on X , then $D/\mu_n \rightarrow X/\mu_n$ is an effective Cartier divisor whose ideal is isomorphic to the \mathcal{O}_X -module $\mathcal{L}^{\otimes -r}$.*

Proof. Let d be the section. Then the closed embedding $D/\mu_n \rightarrow X/\mu_n$ is given by the ideal $(d \cdot \varphi_* \mathcal{O}_X) \cap (\varphi_* \mathcal{O}_X)_0 = d \cdot (\varphi_* \mathcal{O}_X)_{-r \pmod n}$, which is isomorphic to $\mathcal{L}^{\otimes -r}$. \square

3.5.6. Lemma. *Let p be a prime number, and X a variety with a μ_p -action. If $X^{\mu_p} = \emptyset$, then μ_p acts freely on X .*

Proof. We may assume that X is the spectrum of a \mathbb{Z}/p -graded k -algebra A . By assumption, the variety X is covered by the open subschemes $D(f)$ with $f \in A_m$ for $m \in \mathbb{Z}/p - \{0\}$. Thus we may assume that there is $m \in \mathbb{Z}/p - \{0\}$ such that A_m contains an element a_m invertible in A . Since p is prime, it follows that for each $i \in \mathbb{Z}/p$ the set A_i contains an element a_i invertible in A (a power of a_m). Thus $A_i \otimes_{A_0} A_j \rightarrow A_{i+j}$ is an isomorphism for each $i, j \in \mathbb{Z}/p$ (an inverse is given by $a \mapsto a_i \otimes (a_i^{-1}a)$), proving the statement. \square

3.6. Constant finite abelian groups. Let Γ be an ordinary finite abelian group. The group structure on Γ induce a Hopf algebra structure on the k -algebra k^Γ of maps of sets $\Gamma \rightarrow k$ (see e.g. [Wat79, §2.3]). The corresponding algebraic group $\Gamma_k = \text{Spec } k^\Gamma$ is such that $\text{Hom}_k(\text{Spec } R, \Gamma_k) = \Gamma$ for any commutative connected k -algebra R . An action of the ordinary group Γ on a variety X (in the sense of §1) is the same as an action of the algebraic group Γ_k and the notions of equivariant morphisms coincide, so that the closed subschemes X^Γ and X^{Γ_k} are the same (as they represent the same subfunctor).

3.6.1. Lemma. *Let Γ be an ordinary finite abelian group. If the field k contains a root of unity whose order is the exponent of Γ , then the algebraic group Γ_k is finite diagonalisable.*

Proof. Let Γ^* be the ordinary group of group morphisms $\Gamma \rightarrow k^\times$. We prove that the algebraic group Γ_k is isomorphic to $D(\Gamma^*)$. Consider the k -algebra morphism $u: k[\Gamma^*] \rightarrow k^\Gamma$ sending e_χ to $\chi \in \Gamma^* \subset k^\Gamma$. One checks that u is a morphism of Hopf algebras (in fact the subset Γ^* of k^Γ consists of group-like elements). By linear independence of characters [Bou07, V, §6, p.26, Corollaire 1], the morphism

u is injective. By the assumptions on the field k we have $|\Gamma^*| = |\Gamma|$, hence $\dim_k k[\Gamma^*] = \dim_k k^\Gamma$, so that u is an isomorphism. \square

4. THE ROST OPERATION

4.1. The operation ρ . From now on p will be a prime number, and M a cycle module over k such that $p \cdot M = 0$.

4.1.1. Notation. Let X be a variety with a μ_p -action. We endow \mathbb{A}^1 with the μ_p -action of (3.5.4), and consider the μ_p -invariant open subscheme of $X \times \mathbb{A}^1$

$$X^\circ = (X \times \mathbb{A}^1) - (X \times \mathbb{A}^1)^{\mu_p} = (X \times \mathbb{A}^1) - (X^{\mu_p} \times 0).$$

By (3.5.6), the group μ_p acts freely on X° .

If $f: Y \rightarrow X$ is a μ_p -equivariant morphism such that $f^{-1}(X^{\mu_p}) = Y^{\mu_p}$, then there is an induced morphism $f^\circ: Y^\circ \rightarrow X^\circ$.

4.1.2. Definition. Let X be a variety with a μ_p -action. By (3.2.3.ii), the morphism $X^{\mu_p} = X^{\mu_p} \times 0 \rightarrow (X \times \mathbb{A}^1)/\mu_p$ is a closed embedding, and its open complement is X°/μ_p by (3.1.2.ii). We consider the connecting homomorphism

$$\partial_X: A(X^\circ/\mu_p, M) \rightarrow A(X^{\mu_p}, M),$$

the invertible $\mathcal{O}_{X^\circ/\mu_p}$ -module $\mathcal{L} = \mathcal{L}_{X^\circ}$ (see (3.5.1)), the section $\jmath = \jmath_{X^\circ} \in H^0(X^\circ/\mu_p, \mathbb{G}_m/p)$ (see (3.5.2)), and define a morphism

$$\rho_X = \partial_X \circ \{\jmath\} \circ c(-\mathcal{L}): A(X^\circ/\mu_p, M) \rightarrow A(X^{\mu_p}, M).$$

4.1.3. Proposition. *Let $f: Y \rightarrow X$ be a flat μ_p -equivariant morphism such that $f^{-1}(X^{\mu_p}) = Y^{\mu_p}$. Then the morphism $f^\circ/\mu_p: Y^\circ/\mu_p \rightarrow X^\circ/\mu_p$ is flat, and $(f^{\mu_p})^* \circ \rho_X = \rho_Y \circ (f^\circ/\mu_p)^*$.*

Proof. The first statement follows by faithfully flat descent from (3.4.2.iii). We have, as morphisms $A(X^\circ/\mu_p, M) \rightarrow A(Y^{\mu_p}, M)$,

$$\begin{aligned} (f^{\mu_p})^* \circ \rho_X &= (f^{\mu_p})^* \circ \partial_X \circ \{\jmath\} \circ c(-\mathcal{L}) \\ &= \partial_Y \circ (f^\circ/\mu_p)^* \circ \{\jmath\} \circ c(-\mathcal{L}) && \text{by [Ros96, (4.4.2)]} \\ &= \partial_Y \circ \{\jmath\} \circ (f^\circ/\mu_p)^* \circ c(-\mathcal{L}) && \text{by (2.8.3.iii)} \\ &= \partial_Y \circ \{\jmath\} \circ c(-\mathcal{L}) \circ (f^\circ/\mu_p)^* && \text{by (2.7.1.ii)} \\ &= \rho_Y \circ (f^\circ/\mu_p)^*. \end{aligned} \quad \square$$

4.1.4. Corollary. *Let $u: U \rightarrow X$ and $v: V \rightarrow X$ be two μ_p -invariant open immersions such that $U \cup V = X$ and $U^{\mu_p} \cap V^{\mu_p} = \emptyset$. Then*

$$\rho_X = (u^{\mu_p})_* \circ \rho_U \circ u^* + (v^{\mu_p})_* \circ \rho_V \circ v^*.$$

Proof. Indeed X^{μ_p} is the disjoint union of the open subschemes U^{μ_p} and V^{μ_p} , so that $(u^{\mu_p})_* \circ (u^{\mu_p})^* + (v^{\mu_p})_* \circ (v^{\mu_p})^*$ is the identity of $\text{CH}(X^{\mu_p})/p$, and by (4.1.3)

$$\rho_X = ((u^{\mu_p})_* \circ (u^{\mu_p})^* + (v^{\mu_p})_* \circ (v^{\mu_p})^*) \circ \rho_X = (u^{\mu_p})_* \circ \rho_U \circ u^* + (v^{\mu_p})_* \circ \rho_V \circ v^*. \quad \square$$

4.1.5. Proposition. *Let $f: Y \rightarrow X$ be a proper μ_p -equivariant morphism. Then the following diagram commutes.*

$$\begin{array}{ccccc} A(Y^\circ/\mu_p, M) & \xrightarrow{(u/\mu_p)^*} & A(Y'/\mu_p, M) & \xrightarrow{(h/\mu_p)_*} & A(X^\circ/\mu_p, M) \\ \rho_Y \downarrow & & & & \downarrow \rho_X \\ A(Y^{\mu_p}, M) & \xrightarrow{(f^{\mu_p})_*} & & & A(X^{\mu_p}, M) \end{array}$$

Here $Y' = (Y \times \mathbb{A}^1) - (f^{-1}(X^{\mu_p}) \times 0)$, and $u: Y' \rightarrow Y^\circ, h: Y' \rightarrow X^\circ$ are the induced morphisms.

In particular if $f^{-1}(X^{\mu_p}) = Y^{\mu_p}$, then $(f^{\mu_p})_* \circ \rho_Y = \rho_X \circ (f^\circ/\mu_p)_*$.

Proof. Consider the following diagram

$$\begin{array}{ccccc} A(Y^\circ/\mu_p, M) & \xrightarrow{(u/\mu_p)^*} & A(Y'/\mu_p, M) & \xrightarrow{(h/\mu_p)_*} & A(X^\circ/\mu_p, M) \\ \partial_Y \downarrow & & \downarrow & & \downarrow \partial_X \\ A(Y^{\mu_p}, M) & \longrightarrow & A(f^{-1}(X^{\mu_p}), M) & \longrightarrow & A(X^{\mu_p}, M) \end{array}$$

where vertical arrows are connecting homomorphisms, and lower horizontal ones are proper push-forwards (the composite is $(f^{\mu_p})_*$). The square on the right commutes by [Ros96, (4.4.1)] and so does the one on the left by (2.6.1). Thus

$$\begin{aligned} (f^{\mu_p})_* \circ \rho_Y &= (f^{\mu_p})_* \circ \partial_Y \circ \{\mathcal{J}\} \circ c(-\mathcal{L}) \\ &= \partial_X \circ (h/\mu_p)_* \circ (u/\mu_p)^* \circ \{\mathcal{J}\} \circ c(-\mathcal{L}) \\ &= \partial_X \circ \{\mathcal{J}\} \circ (h/\mu_p)_* \circ (u/\mu_p)^* \circ c(-\mathcal{L}) && \text{by (2.8.3)} \\ &= \partial_X \circ \{\mathcal{J}\} \circ c(-\mathcal{L}) \circ (h/\mu_p)_* \circ (u/\mu_p)^* && \text{by (2.7.1)} \\ &= \rho_X \circ (h/\mu_p)_* \circ (u/\mu_p)^*. \end{aligned} \quad \square$$

4.1.6. Lemma. *Let X be a variety with trivial μ_p -action. Then the composite*

$$A(X, M) \xrightarrow{h^*} A(X^\circ/\mu_p, M) \xrightarrow{\rho_X} A(X, M)$$

is the identity, where $h: X^\circ/\mu_p = X \times ((\mathbb{A}^1 - 0)/\mu_p) \rightarrow X$ is the first projection.

Proof. Let $k[t]$ be the coordinate ring of \mathbb{A}^1 . In view of (3.5.4), we see that

$$\rho_X = \partial_X \circ \{\mathcal{J}\} \circ c(-\mathcal{L}) = \partial_X \circ \{t^p\}.$$

Now $0 \rightarrow \mathbb{A}^1/\mu_p$ is the principal Cartier divisor given by the global section t^p , so that by [Ros96, (4.5)] the composite $\partial_X \circ \{t^p\} \circ h^*$ is the identity of $A(X, M)$. \square

Let X be a variety with a μ_p -action. Let D be the deformation variety for the closed embedding $X^{\mu_p} \rightarrow X$; it is defined as the open complement of the strict transform of $X \times 0$ in the blow-up of $X^{\mu_p} \times 0$ in $X \times \mathbb{A}^1$, see [Ros96, (10.4)]. The trivial μ_p -action on \mathbb{A}^1 together with the given μ_p -action on X induce a μ_p -action on D , and D contains the normal cone N (endowed with its natural μ_p -action) as

a μ_p -invariant closed subscheme with open complement $X \times (\mathbb{A}^1 - 0)$. It follows by faithfully flat descent from (3.4.2.iii) (or directly from (3.2.3.ii)) that D°/μ_p contains N°/μ_p as a closed subscheme, with open complement $(X^\circ/\mu_p) \times (\mathbb{A}^1 - 0)$ by (3.1.2.ii). Let

$$\delta: A((X^\circ/\mu_p) \times (\mathbb{A}^1 - 0), M) \rightarrow A(N^\circ/\mu_p, M)$$

be the connecting morphism, $q: (X^\circ/\mu_p) \times (\mathbb{A}^1 - 0) \rightarrow X^\circ/\mu_p$ the first projection, and $t \in H^0(\mathbb{A}^1 - 0, \mathbb{G}_m)$ the section induced by the identification $\mathbb{A}^1 = \text{Spec } k[t]$. We define the morphism

$$\sigma = \delta \circ \{t\} \circ q^*: A(X^\circ/\mu_p, M) \rightarrow A(N^\circ/\mu_p, M).$$

4.1.7. Lemma. *Using the notation just above, we have*

$$\rho_N \circ \sigma = \rho_X: A(X^\circ/\mu_p, M) \rightarrow A(X^{\mu_p}, M).$$

Proof. We view $X^{\mu_p} = X^{\mu_p} \times 0$ as a closed subscheme of $X^{\mu_p} \times \mathbb{A}^1$, and consider the corresponding connecting homomorphism

$$\Delta: A(X^{\mu_p} \times (\mathbb{A}^1 - 0), M) \rightarrow A(X^{\mu_p}, M).$$

Then we have, as morphisms $A(X^\circ/\mu_p, M) \rightarrow A(X^{\mu_p}, M)$,

$$\begin{aligned} \rho_N \circ \sigma &= \partial_N \circ \{j\} \circ c(-\mathcal{L}) \circ \delta \circ \{t\} \circ q^* \\ &= \partial_N \circ \{j\} \circ \delta \circ \{t\} \circ q^* \circ c(-\mathcal{L}) && \text{by (2.7.1)} \\ &= \partial_N \circ \delta \circ \{t\} \circ q^* \circ \{j\} \circ c(-\mathcal{L}) && \text{by (2.8.3)} \\ &= -\Delta \circ \partial_{X \times (\mathbb{A}^1 - 0)} \circ \{t\} \circ q^* \circ \{j\} \circ c(-\mathcal{L}) && \text{by [Ros96, (11.6)]} \\ &= \Delta \circ \{t\} \circ \partial_{X \times (\mathbb{A}^1 - 0)} \circ q^* \circ \{j\} \circ c(-\mathcal{L}) && \text{by [Ros96, (4.3.2)]} \\ &= \Delta \circ \{t\} \circ q^* \circ \partial_X \circ \{j\} \circ c(-\mathcal{L}) && \text{by [Ros96, (4.4.2)]} \\ &= \partial_X \circ \{j\} \circ c(-\mathcal{L}) && \text{by [Ros96, (4.5)]} \\ &= \rho_X. \end{aligned}$$

□

4.2. The degree formula.

4.2.1. Definition. Let X be a variety with a μ_p -action. Taking for M the modulo p Milnor K -theory cycle module in (4.1.2), we define

$$\varrho(X) = \rho_X[X^\circ/\mu_p] \in A(X^{\mu_p}, K_0/p) = \text{CH}(X^{\mu_p})/p,$$

and denote by $\varrho_d(X) \in \text{CH}_d(X^{\mu_p})/p$ its component of degree d .

4.2.2. Proposition (Rost's degree formula). *Let $f: Y \rightarrow X$ be a proper μ_p -equivariant morphism with a degree (see §2.4). Then*

$$(f^{\mu_p})_* \varrho(Y) = \deg f \cdot \varrho(X) \in \text{CH}(X^{\mu_p})/p.$$

Proof. Let $d \in \mathbb{Z}$ be the degree of f . Consider the diagram with cartesian squares (by (3.4.2.iii))

$$\begin{array}{ccccc} Y'/\mu_p & \longleftarrow & Y' & \longrightarrow & Y \\ h/\mu_p \downarrow & & \downarrow h & & \downarrow f \\ X^\circ/\mu_p & \xleftarrow{\varphi_{X^\circ}} & X^\circ & \longrightarrow & X \end{array}$$

The morphism h has degree d by [EKM08, Proposition 49.20 (1)]. The flat pull-back $(\varphi_{X^\circ})^*: \mathcal{Z}(X^\circ/\mu_p) \rightarrow \mathcal{Z}(X^\circ)$ is injective, because $(\varphi_{X^\circ})_* \circ (\varphi_{X^\circ})^* = p \cdot \text{id}$, and $\mathcal{Z}(X^\circ/\mu_p)$ has no p -torsion. Applying once again [EKM08, Proposition 49.20 (1)], we deduce that h/μ_p has degree d . The statement then follows from (4.1.5). \square

4.2.3. Corollary. *Let X be a projective variety without zero-dimensional component. If μ_p acts on X , then $\deg \varrho(X) = 0 \in \mathbb{F}_p$.*

Proof. Apply (4.2.2) to the proper μ_p -equivariant morphism $X \rightarrow \text{Spec } k$, which has degree zero. \square

4.3. Computation of the first term.

4.3.1. Lemma. *Let \mathcal{V} be a finite-dimensional μ_p -representation over k such that $\mathcal{V}_0 = 0$, and Y an equidimensional variety with trivial μ_p -action. Consider the variety $V = \text{Spec}(\text{Sym}_k \mathcal{V}^\vee)$ with its induced μ_p -action. Then the cycle class $\varrho(Y \times V) \in \text{CH}(Y)/p$ is a non-zero multiple of $[Y]$.*

Proof. By (4.1.3), we may assume that $Y = \text{Spec } k$. We proceed by induction on the dimension d of \mathcal{V} . The statement follows from (4.1.6) if $d = 0$. If $d > 0$, we may decompose the μ_p -representation \mathcal{V} as $\mathcal{W} \oplus \mathcal{L}$, with \mathcal{L} one-dimensional. Since $\mathcal{V}_0 = 0$, we must have $\mathcal{L} = \mathcal{L}_r$ for some $r \in \mathbb{Z}/p - \{0\}$. The closed embedding $i: W = \text{Spec}(\text{Sym}_k \mathcal{W}^\vee) \rightarrow V$ is a principal effective Cartier divisor given by a section of weight $-r$ (see (3.2.5)), hence the same is true for the closed embedding i° . By (3.5.5), the closed embedding i°/μ_p is an effective Cartier divisor whose ideal is isomorphic to $\mathcal{L}^{\otimes r}$. Thus we have in $\text{CH}(\text{Spec } k)/p = \mathbb{F}_p$,

$$\begin{aligned} \varrho(W) &= (i^{\mu_p})_* \circ \rho_W[W^\circ/\mu_p] && \text{since } i^{\mu_p} = \text{id}_{\text{Spec } k} \\ &= \rho_V \circ (i^\circ/\mu_p)_*[W^\circ/\mu_p] && \text{by (4.1.5)} \\ &= \rho_V \circ c_1(\mathcal{L}^{\otimes -r})[V^\circ/\mu_p] && \text{by [Ful98, (3.2) (f)]} \\ &= -r \cdot \rho_V \circ c_1(\mathcal{L})[V^\circ/\mu_p] && \text{by [Ful98, (2.5) (e)]} \\ &= r \cdot \varrho(V) && \text{since } \varrho_d(V) \in \text{CH}_d(\text{Spec } k)/p = 0. \end{aligned}$$

We conclude using the induction hypothesis. \square

4.3.2. Lemma. *Let X be an equidimensional variety with a μ_p -action. Let N be the normal cone of the closed embedding $X^{\mu_p} \rightarrow X$. Then $\varrho(X) = \varrho(N) \in \text{CH}(X^{\mu_p})/p$.*

Proof. We use the notation given directly above (4.1.7). Consider the commutative diagram

$$\begin{array}{ccccc} N^\circ & \longrightarrow & N^\circ/\mu_p & \xrightarrow{n} & 0 \\ \downarrow & & \downarrow & & \downarrow \\ D^\circ & \longrightarrow & D^\circ/\mu_p & \longrightarrow & \mathbb{A}^1 \end{array}$$

The exterior square is cartesian, and so is the left one by (3.4.2.iii). The lower left horizontal arrow is faithfully flat, hence by descent the square on the right is cartesian. Let $\delta_0: A(\mathbb{A}^1 - 0, K_1/p) \rightarrow \mathrm{CH}(0)/p$ be the connecting homomorphism associated with closed embedding $0 \rightarrow \mathbb{A}^1$. The variety X°/μ_p is equidimensional, because the morphism $X^\circ \rightarrow X^\circ/\mu_p$ is finite and surjective and the variety X° is equidimensional (being open in $X \times \mathbb{A}^1$). Therefore the second projection $x: X^\circ/\mu_p \times (\mathbb{A}^1 - 0) \rightarrow \mathbb{A}^1 - 0$ is flat of constant relative dimension. We have in $\mathrm{CH}(N^\circ/\mu_p)/p$

$$\begin{aligned} \sigma[X^\circ/\mu_p] &= \delta \circ \{t\}[X^\circ/\mu_p \times (\mathbb{A}^1 - 0)] && \text{by [Ful98, Lemma 1.7.1]} \\ &= \delta \circ \{t\} \circ x^*[\mathbb{A}^1 - 0] && \text{by [Ful98, Lemma 1.7.1]} \\ &= \delta \circ x^* \circ \{t\}[\mathbb{A}^1 - 0] && \text{by [Ros96, (4.3.1)]} \\ &= n^* \circ \delta_0 \circ \{t\}[\mathbb{A}^1 - 0] && \text{by [Ros96, (4.4.2)]} \\ &= n^*[0] = [N^\circ/\mu_p] && \text{by [Ros96, (4.5)].} \end{aligned}$$

Thus the statement follows from (4.1.7). \square

4.3.3. Proposition. *Let X be a variety with a μ_p -action. Assume that X^{μ_p} is irreducible of dimension d and that X is regular at the generic point of X^{μ_p} . Then*

$$\varrho_d(X) \neq 0 \in \mathrm{CH}_d(X^{\mu_p})/p = \mathbb{F}_p.$$

Proof. Let U be a μ_p -invariant open subscheme of X meeting X^{μ_p} . Then the restriction morphism $\mathrm{CH}_d(X^{\mu_p})/p \rightarrow \mathrm{CH}_d(U^{\mu_p})/p$ is injective and sends $\varrho_d(X)$ to $\varrho_d(U)$ by (4.1.3). Therefore we may replace X by U . Let R be a connected regular open subscheme of X meeting X^{μ_p} . The μ_p -invariant open subscheme R' of R produced by (3.1.1.ii) meets X^{μ_p} by (3.1.1.ii.b). Replacing X with R' , we may assume that X is regular and connected. Then X^{μ_p} is also regular by (3.3.2). Let $q: N \rightarrow X^{\mu_p}$ be the normal bundle of the regular closed embedding $X^{\mu_p} \rightarrow X$, and \mathcal{N} its $\mathcal{O}_{X^{\mu_p}}$ -module of sections. The μ_p -action on X induces a decomposition $\mathcal{N} = \bigoplus_{i \in \mathbb{Z}/p} \mathcal{N}_i$ as $\mathcal{O}_{X^{\mu_p}}$ -modules. Further shrinking X , we may assume that each $\mathcal{O}_{X^{\mu_p}}$ -module \mathcal{N}_i is free. Then there is μ_p -representation \mathcal{V} over k and a μ_p -equivariant isomorphism $\mathcal{N} = \mathcal{V} \otimes_k \mathcal{O}_{X^{\mu_p}}$. Since $N^{\mu_p} = X^{\mu_p}$ by (3.3.3), we have $\mathcal{N}_0 = \mathcal{V}_0 = 0$. By (4.3.1), there is an element $u \in (\mathbb{F}_p)^\times$ such that $\varrho(N) = u \cdot [X^{\mu_p}]$ in $\mathrm{CH}(X^{\mu_p})/p$. We conclude using (4.3.2). \square

4.3.4. Remark. The element $u \in (\mathbb{F}_p)^\times$ such that $\varrho_d(X) = u \cdot [X^{\mu_p}]$ in (4.3.3) can be explicitly computed in terms of the μ_p -action on the normal bundle of X^{μ_p} in X . Indeed, its fiber at the generic point of X^{μ_p} is a μ_p -representation \mathcal{V} over the field $K = k(X^{\mu_p})$, and it follows from the proof of (4.1.6) that

$$u = \prod_{i \in \mathbb{Z}/p - \{0\}} i^{-\dim_K \mathcal{V}_i} \in (\mathbb{F}_p)^\times.$$

5. EQUIVARIANCE OF THE OPERATION

5.1. Equivariant Cycles. When G is a finite constant group acting on a variety X , one may define the set of G -equivariant cycles as the equaliser of the two pull-back maps $\mathcal{Z}(X) \rightarrow \mathcal{Z}(G \times X)$. This definition is however inappropriate when G is arbitrary (consider the action of μ_p on itself in characteristic p), and we will instead use the following definition:

5.1.1. Definition. Let G be an algebraic group acting on a variety X . We denote by $\mathcal{Z}_G(X) \subset \mathcal{Z}(X)$ the subgroup generated by classes of (possibly non-integral) G -invariant closed subschemes of X .

5.1.2. Lemma. *Let G be an algebraic group acting on a variety X , and U_i a family of G -invariant open subschemes covering X . Then a cycle in $\mathcal{Z}(X)$ belongs to $\mathcal{Z}_G(X)$ if and only if its restriction to $\mathcal{Z}(U_i)$ belongs to $\mathcal{Z}_G(U_i)$ for every i .*

Proof. Let $u_i: U_i \rightarrow X$ be the open immersions. Clearly $u_i^* \mathcal{Z}_G(X) \subset \mathcal{Z}_G(U_i)$. Conversely, we prove by induction on s that a cycle $z = \sum_{\alpha=1}^s n_\alpha [Z_\alpha] \in \mathcal{Z}(X)$ (with $n_\alpha \in \mathbb{Z}$ and Z_α integral closed subschemes of X) belongs to $\mathcal{Z}_G(X)$ as soon as $u_i^*(z) \in \mathcal{Z}_G(U_i)$ for each i . Assuming $s \geq 1$, let j be such that $Z_1 \cap U_j \neq \emptyset$, and consider the cycle $z' = z - (u_j)_* \circ u_j^*(z) \in \mathcal{Z}(X)$. Since the scheme-theoretic closure of a G -invariant closed subscheme of U_j is a G -invariant closed subscheme of X , it follows that $(u_j)_* \mathcal{Z}_G(U_j) \subset \mathcal{Z}_G(X)$, so that

$$(5.1.a) \quad (u_j)_* \circ u_j^*(z) \in \mathcal{Z}_G(X),$$

and therefore $u_i^*(z') = u_i^*(z) - u_i^* \circ (u_j)_* \circ u_j^*(z) \in \mathcal{Z}_G(U_i)$ for each i . But $z' = \sum_\alpha n_\alpha [Z_\alpha]$, where α runs over the elements of $\{1, \dots, s\}$ such that $Z_\alpha \cap U_j = \emptyset$. Since $Z_1 \cap U_j \neq \emptyset$, we may apply the induction hypothesis to the cycle z' , and deduce that $z' \in \mathcal{Z}_G(X)$, which in view of (5.1.a) implies that $z = z' + (u_j)_* \circ u_j^*(z) \in \mathcal{Z}_G(X)$. \square

5.1.3. Definition. A non-empty variety with a G -action in which every non-empty G -invariant open subscheme is dense will be called *G -irreducible*.

5.1.4. Lemma. *Let G be an algebraic group acting on a variety X . Then the group $\mathcal{Z}_G(X)$ is generated by classes of G -irreducible closed subschemes of X .*

Proof. It will suffice to prove that for any non-empty variety Y with a G -action, the class $[Y] \in \mathcal{Z}(Y)$ may be written as the sum of classes of G -irreducible closed

subschemes of Y . We proceed by induction on the number of irreducible components of Y . We assume that Y is non-empty and not G -irreducible, since the statement clearly holds otherwise. Then we may find a non-empty and non-dense G -invariant open subscheme of Y . Let Y_0 be its scheme-theoretic closure in Y , and Y_1 the scheme-theoretic closure of $Y - Y_0$ in Y . Then the set of generic points of Y is the disjoint union of the sets of generic points of Y_0 and Y_1 , and the multiplicities coincide. It follows that $[Y] = [Y_0] + [Y_1]$ in $\mathcal{Z}(Y)$. Since the G -invariant closed subschemes Y_0 and Y_1 each contain at least one generic point of Y , we may conclude by induction. \square

5.1.5. Lemma. *Let G be a finite algebraic group and X a G -irreducible variety. Then X is equidimensional.*

Proof. Let Z be the union of the irreducible components of X whose dimension is strictly smaller than $\dim X$, and let $U = X - Z$. The G -equivariant open subscheme U' of X produced by (3.1.1.ii) is non-empty since $\dim(X - U') = \dim Z < \dim X$. Thus U' , hence also U , is dense in X . This implies that Z contains no generic point of X , hence $Z = \emptyset$. \square

5.1.6. Lemma. *Let G be an algebraic group, and $i: Y \rightarrow X$ the embedding of a G -invariant closed subscheme. For any $\alpha \in \mathcal{Z}_G(X)$, the cycle $i_*(\alpha|_Y) \in \mathcal{Z}(X)$ is a \mathbb{Z} -linear combination of classes of G -invariant closed subschemes of X whose support is contained in the support of Y .*

Proof. By (5.1.4) we may assume that $\alpha = [X]$ and that X is G -irreducible. Then either i is surjective and $i_*([X]|_Y) = [X]$, or Y contains no generic point of X and $[X]|_Y = 0$. In either case, the statement is true. \square

5.2. Representing $\varrho(X)$ by equivariant cycles.

5.2.1. Lemma. *Let G be a finite diagonalisable group acting on a variety X , and $u: U \rightarrow X$ the embedding of an equidimensional G -invariant open subscheme. Let $a \in H^0(U, \mathbb{G}_m)$ be of weight $g \in \widehat{G}$ (as a section of \mathcal{O}_U , see (3.2.5)). Then $[U]$ is mapped to an element of $\mathcal{Z}_G(X)$ under the composite*

$$\mathcal{Z}(U) \xrightarrow{\{a\}} C(U, K_1) \xrightarrow{u_*} C(X, K_1) \xrightarrow{d} \mathcal{Z}(X).$$

Proof. Replacing X by the scheme-theoretic closure of U , we may assume that U contains all associated points of X and that X is equidimensional. In view of (5.1.2), we may replace X with a cover by G -invariant open subschemes, and thus assume that X is the spectrum of a \widehat{G} -graded k -algebra A . We are going to find an element $f \in A_0$, which is a nonzerodivisor in A and is such that $D(f) \subset U$. First observe that U admits a G -invariant closed complement in X (let R be the reduced closed complement of U/G in X/G , and take $R \times_{X/G} X$); let $J \subset A$ be its ideal. We claim that J_0 is contained in no associated prime of A . Indeed let \mathfrak{p} be a prime of A such that $J_0 \subset \mathfrak{p}$. If $x \in J_g$ for some $g \in \widehat{G}$, then $x^{|G|} \in J_0 \subset \mathfrak{p}$,

hence $x \in \mathfrak{p}$. Since J is a graded ideal of A , it follows that $J \subset \mathfrak{p}$. Thus \mathfrak{p} cannot be an associated prime of A (as they all lie in U), which proves the claim. By prime avoidance, we may find $f \in J_0$ such that f is in none of the primes $\mathfrak{p} \cap A_0$ of A_0 for \mathfrak{p} an associated prime of A ; then f has the required properties.

Since $D(f)$ is dense in X (being the complement of an effective Cartier divisor) and contained in U , we may replace U by $D(f)$ while proving the lemma. Then $a = b/f^n \in A[1/f]^\times$ for some integer $n \geq 0$ and $b \in A$. We view A a subalgebra of $A[1/f]$ (as f is a nonzerodivisor in A). Then b is nonzerodivisor in A , as is any element of $A[1/f]^\times \cap A$. Since $f \in A_0$, the A -module $A[1/f]$ is graded, and $b \in A_g = (A[1/f])_g \cap A$. Then, by [Ful98, Lemma 1.7.2]

$$d \circ u_* \circ \{a\}[U] = d \circ u_* \circ \{b\}[U] - n \cdot d \circ u_* \circ \{f\}[U] = [V(b)] - n \cdot [V(f)],$$

where $V(b)$, resp. $V(f)$, denotes the closed subscheme of X defined by the ideal of A generated by the element b , resp. f . Since b and f are homogeneous elements of the \widehat{G} -graded algebra A , it follows that these closed subschemes are G -invariant. This concludes the proof. \square

5.2.2. Lemma. *Let G be a finite diagonalisable group containing μ_n , and X a variety with a G -action. Assume that μ_n acts freely on X (see (3.4.1)). Then:*

- (i) *The G -action on X induces a G -action on X/μ_n and a G -equivariant structure on the \mathcal{O}_{X/μ_n} -module \mathcal{L} defined in (3.5.1).*
- (ii) *The G -equivariant \mathcal{O}_{X/μ_n} -module \mathcal{L} is locally (in the G -equivariant Zariski topology) isomorphic to the pull-back of a G -representation over k .*
- (iii) *The image of $\mathcal{Z}_G(X/\mu_n)$ in $\mathrm{CH}(X/\mu_n)$ is stable under $c_1(\mathcal{L})$.*
- (iv) *Let $u: X \rightarrow \overline{X}$ be a G -invariant open embedding, and consider the section $s \in H^0(X/\mu_n, \mathbb{G}_m/n)$ defined in (3.5.2). Then the composite*

$$\mathcal{Z}(X/\mu_n)/n \xrightarrow{\{s\}} C(X/\mu_n, K_1/n) \xrightarrow{u_*} C(\overline{X}/\mu_n, K_1/n) \xrightarrow{d} \mathcal{Z}(\overline{X}/\mu_n)/n$$

maps the image of $\mathcal{Z}_G(X/\mu_n)$ into the image of $\mathcal{Z}_G(\overline{X}/\mu_n)$.

Proof. (i): The morphism $X \xrightarrow{\varphi} X/G$ is μ_n -equivariant, hence factors as $X \xrightarrow{\psi} X/\mu_n \xrightarrow{\lambda} X/G$. For $i \in \mathbb{Z}/n$, we have $\lambda_*((\psi_* \mathcal{O}_X)_i) = \bigoplus_{\alpha(g)=i} (\varphi_* \mathcal{O}_X)_g$, where $\alpha: \widehat{G} \rightarrow \widehat{\mu_n} = \mathbb{Z}/n$ is induced by the inclusion $\mu_n \subset G$. The first part of (i) follows from the case $i = 0$, (3.2.1) and (3.2.2). The second part follows from the case $i = 1$.

(ii): We may assume that X is the spectrum of a \widehat{G} -graded k -algebra A , and that the A_0 -module A_1 is free with basis a (here $0, 1 \in \mathbb{Z}/n = \widehat{\mu_n}$). For $g \in \widehat{G}$ mapping to $1 \in \widehat{\mu_n} = \mathbb{Z}/n$, denote by a_g the component of a in A_g , and by U_g the support of the section $a_g \in H^0(X/\mu_n, \mathcal{L})$. Then the variety X/μ_n is covered by the G -invariant open subschemes U_g , and $\mathcal{L}|_{U_g}$ is the pull-back of the one-dimensional G -representation over k having a non-zero component in degree g .

(iii): First observe that any G -invariant closed subscheme of X/μ_n is of the form Y/μ_n for some G -invariant closed subscheme Y of X (indeed if T is a G -invariant closed subscheme of X/μ_n , then the morphism $(T \times_{X/\mu_n} X)/\mu_n \rightarrow T$ is an isomorphism by (3.4.2.iii)). Thus in view of (5.1.4), it will suffice to prove that $c_1(\mathcal{L})[X/\mu_n] \in \text{CH}(X/\mu_n)$ is represented by a cycle in $\mathcal{Z}_G(X/\mu_n)$ under the assumption that X/μ_n is G -irreducible, and therefore also equidimensional by (5.1.5). By (ii), we may find a cover of X/μ_n by G -invariant open subschemes V_i , and for each i a nowhere vanishing section l_i of $\mathcal{L}|_{V_i}$ of weight $g_i \in \widehat{G}$. Let us fix an index a such that $V_a \neq \emptyset$, and thus V_a is dense in X/μ_n . For each i , let $s_i \in H^0(V_i \cap V_a, \mathbb{G}_m)$ be the element such that $l_a = s_i l_i$ as a section of \mathcal{L} on $V_i \cap V_a$. The section s_i of $\mathcal{O}_{V_i \cap V_a}$ has weight $g_a - g_i \in \widehat{G}$. By (2.7.2) and (5.2.1), the cycle class $c_1(\mathcal{L})[X/\mu_n] \in \text{CH}(X/\mu_n)$ is represented by a cycle in $\mathcal{Z}(X/\mu_n)$ whose restriction to each V_i belongs to $\mathcal{Z}_G(V_i)$. We conclude using (5.1.2).

(iv): Let T be a G -irreducible closed subscheme of X/μ_n , and $P = T \times_{X/\mu_n} X$. Then the morphism $P/\mu_n \rightarrow T$ is an isomorphism by (3.4.2.iii). Let \overline{P} be the scheme-theoretic closure of P in \overline{X} . In view of (5.1.4), we may replace the open embedding $X \rightarrow \overline{X}$ with $P \rightarrow \overline{P}$, and thus assume that X/μ_n is G -irreducible, and therefore also equidimensional by (5.1.5). By (ii), we may find a dense G -invariant open embedding $v: V \rightarrow X/\mu_n$ and a nowhere vanishing section l of $\mathcal{L}|_V$ of weight $g \in \widehat{G}$. The image $b \in H^0(V, \mathbb{G}_m)$ of $l^{\otimes n}$ under the morphism $\mathcal{L}^{\otimes n} \rightarrow \mathcal{O}_{X/\mu_n}$ (induced by the multiplication of \mathcal{O}_X) has weight $n \cdot g$ as a section of \mathcal{O}_V , and is a lifting of $s|_V \in H^0(V, \mathbb{G}_m/n)$. Thus by (2.8.3.ii)

$$d \circ u_* \circ \{s\}[X/\mu_n] = d \circ u_* \circ \{s\} \circ v_*[V] = d \circ (u \circ v)_* \circ \{s|_V\}[V] \in \mathcal{Z}(\overline{X}/\mu_n)/n$$

is the class modulo n of the cycle $d \circ (u \circ v)_* \circ \{b\}[V] \in \mathcal{Z}(\overline{X}/\mu_n)$, which belongs to $\mathcal{Z}_G(\overline{X}/\mu_n)$ by (5.2.1). \square

5.2.3. Proposition. *Let G be a finite diagonalisable p -group containing μ_p . Let X be a variety with a G -action such that X^{μ_p} is projective and that $X^G = \emptyset$. Then $\deg \varrho(X) = 0 \in \mathbb{F}_p$ (see (4.2.1)).*

Proof. By (5.2.2.iii), the cycle class $c(-\mathcal{L})[X^\circ/\mu_p] \in \text{CH}(X^\circ/\mu_p)$ is represented by some cycle $\alpha \in \mathcal{Z}_G(X^\circ/\mu_p)$. Therefore the modulo p cycle class

$$\varrho(X) = \partial_X \circ \{s\} \circ c(-\mathcal{L})[X^\circ/\mu_p] \in \text{CH}(X^{\mu_p})/p$$

is represented by the element $\beta \in \mathcal{Z}(X^{\mu_p})/p$ such that

$$i_*(\beta) = d \circ u_* \circ \{s\}(\alpha) \in \mathcal{Z}(X/\mu_p)/p$$

where $u: X^\circ/\mu_p \rightarrow X/\mu_p$ is the open embedding, and $i: X^{\mu_p} \rightarrow X/\mu_p$ the closed embedding. It follows from (5.2.2.iv) that $i_*(\beta)$ is the class modulo p of a cycle $\gamma \in \mathcal{Z}_G(X/\mu_p)$. Thus $\beta = i_*(\beta)|_{X^{\mu_p}}$ is the class modulo p of the cycle $\delta = \gamma|_{X^{\mu_p}} \in \mathcal{Z}(X^{\mu_p})$, and to conclude the proof it will suffice to show that $\deg \delta = \deg \circ i_*(\gamma|_{X^{\mu_p}})$ is divisible by p . To do so, it suffices by (5.1.6) to prove that $\deg[Y] \in p\mathbb{Z}$ when Y is a G -invariant closed subscheme of X/μ_p supported on

X^{μ_p} . By (5.2.4) below it will suffice to prove that $Y^G = \emptyset$ for such Y . But for any field extension L/k , the subset $Y^G(L) \subset (X/\mu_p)(L)$ is contained in

$$((X/\mu_p)^G)(L) \cap (X^{\mu_p})(L) = ((X^{\mu_p})^G)(L) = X^G(L) = \emptyset. \quad \square$$

5.2.4. Lemma. *Let G be a finite diagonalisable p -group. Let X be a variety with a G -action such that $X^G = \emptyset$. Then $\deg[X]$ is divisible by p .*

Proof. Let X_0 be the union of the zero-dimensional connected components of X . Then $U = X - X_0$ is open and closed in X , and the G -invariant open subscheme U' produced by (3.1.1.ii) is closed and satisfies $\dim(X - U') = \dim X_0 \leq 0$. It follows that X_0 is the open complement of U' in X , and is therefore G -invariant. Excluding the trivial case $X_0 = \emptyset$ and replacing X with X_0 , we may assume that $\dim X = 0$, hence that X is the spectrum of a \widehat{G} -graded k -algebra A . Replacing X with a cover by G -invariant open (and closed) subschemes, we may assume that for some $g \in \widehat{G} - \{0\}$ the subset A_g contains an element $a \in A^\times$. Let C be the subgroup of \widehat{G} generated by g . For any $h \in \widehat{G}$, multiplication with a induces a k -linear isomorphism $A_h \rightarrow A_{g+h}$, hence $\dim_k A_h$ depends only on the class of h in \widehat{G}/C . Since the order of C is divisible by p , so is for any $E \in \widehat{G}/C$ the dimension of the k -vector space $\bigoplus_{e \in E} A_e$. Since A is the direct sum of those vector spaces, it follows that $\dim_k A$ is divisible by p . \square

6. ISOLATED FIXED POINTS

6.5. Theorem. *Let X be a projective variety without connected component of dimension zero, and G a finite diagonalisable p -group acting on X . Then the set underlying X^G cannot be a single regular closed point of X of degree prime to p .*

Proof. We proceed by induction on $|G|$. If G is trivial, then $X^G = X$ has no isolated point, and the statement is proved. When G is non-trivial, it contains a subgroup isomorphic to μ_p (because the ordinary p -group \widehat{G} admits a quotient isomorphic to \mathbb{Z}/p). We assume that the set underlying the closed subscheme X^G is a single regular point x of X of degree prime to p . The closed subscheme X^{μ_p} of X is G -invariant because G is commutative. Let Y be the connected component of X^{μ_p} containing the subscheme X^G , and $Y' \subset Y$ the G -invariant open subscheme of (3.1.1). Then Y' is closed in X^{μ_p} by (3.1.1.ii.a), and moreover $x \in Y'$ by (3.1.1.ii.b), so that $Y' = Y$. Thus Y is a G -invariant open subscheme of X^{μ_p} , and so is $Z = X^{\mu_p} - Y$. Since the point x of X is regular, it is contained in a regular open subscheme R of X . Replacing R with R' , we may assume by (3.1.1.ii.b) that R is G -invariant. It follows from (3.3.2) that the variety R^{μ_p} , hence also its open subscheme $R^{\mu_p} \cap Y$ is regular. Thus x (being contained in the open subscheme $R^{\mu_p} \cap Y$) is a regular closed point of Y . The induction hypothesis applied to the action of $G/\mu_p = D(\ker(\widehat{G} \rightarrow \widehat{\mu}_p))$ on Y shows that $\dim Y = 0$, which implies that $Y = X^G$. Thus we have obtained a G -equivariant decomposition as disjoint open subschemes $X^{\mu_p} = X^G \sqcup Z$.

The fixed loci $(X - Z)^{\mu_p} = X^G$ and $(X - X^G)^{\mu_p} = Z$ are projective. Since X is projective, we have by (4.2.3) and (4.1.4)

$$0 = \deg \varrho(X) = \deg \varrho(X - Z) + \deg \varrho(X - X^G) \in \mathbb{F}_p.$$

Since $\varrho(X - Z) \neq 0 \in \mathrm{CH}(X^G)/p$ by (4.3.3) and x is a point of degree prime to p , we deduce that $\deg \varrho(X - Z) \neq 0 \in \mathbb{F}_p$. Thus $\deg \varrho(X - X^G) \neq 0 \in \mathbb{F}_p$, in contradiction with (5.2.3). \square

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MATHEMATISCHES INSTITUT, LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN, THERESIEN-
STR. 39, D-80333 MÜNCHEN, GERMANY
E-mail address: olivier.hauton at gmail.com